B.Sc part II(II), paper — III Diff calculus 18-07-2020

Tangents and normals problems

Q.1. Tangent at any point of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ meets the ~~curve~~ co-ordinate axes in A and B. Show that the locus of the point with (OA, OB) as co-ordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Soln: We know that parametric eqns of the curve are

$$x=a\cos^3\theta,\quad y=b\sin^3\theta.$$

Differentiating with respect to θ

$$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta,\quad \frac{dy}{d\theta}=3b\sin^2\theta\cos\theta$$

$$\therefore \frac{dy}{dx}=\frac{-\cancel{3}b\sin^2\theta\cos\theta}{\cancel{3}a\cos^2\theta\sin\theta}=-\frac{b}{a}\frac{\sin\theta}{\cos\theta}$$

$\therefore$ equation of the tangent at (x,y) is

$$Y-y=-\frac{b\sin\theta}{a\cos\theta}(X-x)$$

$$\Rightarrow Y-b\sin^3\theta=-\frac{b}{a}\frac{\sin\theta}{\cos\theta}(X-a\cos^3\theta)$$

$$\Rightarrow a\cos\theta.Y+b\sin\theta.X=ab\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta)$$

$$=ab\sin\theta\cos\theta$$

$$\Rightarrow \frac{X}{a\cos\theta}+\frac{Y}{b\sin\theta}=1$$

$\therefore$ OA $=a\cos\theta$ and OB $=b\sin\theta$.

So, the locus of $(a\cos\theta, b\sin\theta)$. Let $\alpha=a\cos\theta$, $\beta=b\sin\theta$

$$\therefore \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1$$

$\therefore$ Locus of (α,β) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Proved.

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Q.2: Show that the portion of the tangent to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ which is intercepted between the axes, is of constant length and find the area of the portion included between the axes and the tangent.

Soln: The equation of the tangent at (x, y) is

$$(X-x)f_x+(Y-y)f_y=0$$

The eqn of the curve is

$$f(x,y)=x^{\frac{2}{3}}+y^{\frac{2}{3}}-a^{\frac{2}{3}}=0$$

$$\therefore f_x=\frac{2}{3}x^{-\frac{1}{3}},\ f_y=\frac{2}{3}y^{-\frac{1}{3}}$$

$\therefore$ The eqn of the tangent becomes

$$X\frac{2}{3}x^{-\frac{1}{3}}+Y\frac{2}{3}y^{-\frac{1}{3}}=x\cdot\frac{2}{3}x^{-\frac{1}{3}}+y\cdot\frac{2}{3}y^{-\frac{1}{3}}$$

$$=\frac{2}{3}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=\frac{2}{3}a^{\frac{2}{3}}$$

$$\therefore \frac{X}{x^{1/3}}+\frac{Y}{y^{\frac{1}{3}}}=a^{\frac{2}{3}}$$

$$\Rightarrow \frac{X}{a^{\frac{2}{3}}x^{1/3}}+\frac{Y}{a^{\frac{2}{3}}y^{\frac{1}{3}}}=1$$

$\therefore$ intercepts $OA=a^{\frac{2}{3}}x^{\frac{1}{3}}$ and $OB=a^{\frac{2}{3}}y^{\frac{1}{3}}$

$\therefore$ Portion of the tangent intercepted between the axes

$$=AB=\sqrt{OA^2+OB^2}=\sqrt{a^{\frac{4}{3}}x^{\frac{2}{3}}+a^{\frac{4}{3}}y^{\frac{2}{3}}}=a^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$$

$$=a^{\frac{2}{3}}\cdot a^{\frac{1}{3}}=a$$

which is constant.

Area of the portion included between the axes and the tangent = Area of the ΔAOB

$$=\frac{1}{2}\cdot OA\cdot OB=\frac{1}{2}a^{\frac{2}{3}}\cdot x^{1/3}\cdot a^{\frac{2}{3}}\cdot y^{\frac{1}{3}}=\frac{1}{2}a^{\frac{4}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$$

Proved.